FOURIER ANALYSIS

We live in a time-domain world. We think in the time domain and make our observations of occurrences based on how they behave as a function of time. There exists another domain in which we can likewise analyze how a system or signal behaves. This is the frequency domain and since we do not think in the frequency domain, we have some initial difficulty trying to visualize system and signal behavior in this domain.

Why would we want to go to the trouble to analyze a system in the frequency domain if we can adequately analyze it in the time domain? The answer is that it is usually computationally easier to make the transformation into the frequency domain, perform the analysis, and transform back into the time domain with the identical answer, than to perform the analysis in the time domain alone.

When we transform a time domain signal into the frequency domain, we are simply determining the frequency, or spectral, content of the signal. There is a one-to-one correspondence between the time domain signal and its frequency domain counterpart. Therefore, the transformation is completely reversible, i.e., the time domain signal can be recovered from its frequency domain representation. We'll begin by discussing the representation of a time domain function in terms of other time domain functions.

C.1 REPRESENTING x(t) by OTHER TIME DOMAIN FUNCTIONS

How do we represent one function in terms of another? There are probably an infinite number of ways to accomplish this, but for mathematical convenience we will let a signal, say x(t), be represented by a linear combination of basis functions, $\phi_n(t)$, where $0 \le n \le N$. Using this notation,

$$x(t) = \sum_{n=0}^{N} a_n \phi_n(t) \tag{C-1}$$

where the a_n values are the linear coefficients multiplied by the basis functions and N can theoretically be infinity. One property that is desired of a set of basis functions is *finality of coefficients*. This property allows us to determine the individual a_n coefficients without needing to know any of the other coefficients. In other words, N can be increased (for greater accuracy for example) without changing the value of the coefficients already determined. In order to have finality of coefficients, the basis functions must be orthogonal over the time interval for which our representation is to be valid.

The condition of orthogonality of basis functions requires that on the interval $t_2 - t_1$, for all k,

$$\int_{t_1}^{t_2} \phi_n(t)\ \phi^*{}_k(t)\ dt = \begin{cases} 0, & k \neq n \\ \lambda_k, & k = n \end{cases} \tag{C-2}$$

where $\phi^*{}_k(t)$ is the complex conjugate of $\phi_k(t)$ and λ_k is real.

The convenient set of orthogonal basis functions used by Fourier is the sine and cosine. We will presently see that they are indeed orthogonal by integrating them as in Equation C-2.

C.2 FOURIER SERIES

C.2.1 Trigonometric Form of Fourier Series

We can represent periodic signals by a series of linear combinations of sines and cosines. Recall that a signal, x(t), is periodic with period T_0 if

$$x(t + T_0) = x(t). \tag{C-3}$$

The periodic signal x(t) has a fundamental frequency (or 1st harmonic) of $f_0 = 1/T_0$. The second harmonic is $2f_0$, or in general the nth harmonic of the signal is nf_0. For example, if x(t) has a period of 0.1 second, then $f_0 = 10$ Hertz. The second harmonic is then 20 Hz, etc. If n is even, then the harmonic is even and if odd then the harmonic is odd. A signal is composed of its basis functions each containing the fundamental frequency and a linear combination of its harmonics. This combination of sinusoidal basis functions is the Fourier series.

If a signal, x(t), satisfies the Dirichlet conditions (described in Section C.2.7) it can be completely defined by

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \tag{C-4}$$

which is the *trigonometric Fourier series* for x(t). The constants a_0, a_n, and b_n are the Fourier coefficients and are the only values that must be calculated to determine the Fourier series.

The angular frequency ω_0 is found as

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}. \tag{C-5}$$

To calculate these coefficients, we can integrate Equation C-4 over one period, taking advantage of the fact that sinusoids (being orthogonal functions), and products of sinusoids, (with two exceptions) integrate to zero over the period. For arbitrary t_0 and constants m and n,

$$\int_{t_0}^{t_0+T_0} \sin(n\omega_0 t)\, dt = 0 \tag{C-6}$$

$$\int_{t_0}^{t_0+T_0} \cos(n\omega_0 t)\, dt = 0 \tag{C-7}$$

$$\int_{t_0}^{t_0+T_0} \sin(n\omega_0 t)\cos(m\omega_0 t)\, dt = 0 \tag{C-8}$$

$$\int_{t_0}^{t_0+T_0} \sin(n\omega_0 t)\sin(m\omega_0 t)\, dt = \begin{cases} 0, & n \neq m \\ T_0/2, & n = m \end{cases} \tag{C-9}$$

and

$$\int_{t_0}^{t_0+T_0} \cos(n\omega_0 t)\cos(m\omega_0 t)\, dt = \begin{cases} 0, & n \neq m \\ T_0/2, & n = m \end{cases} \tag{C-10}$$

The significance of these equations is that if we integrate over one period a cosine times a sine, or a cosine times a cosine, or a sine time a sine, the result is zero. This is true except for the two cases where we are integrating either the cosine squared or the sine squared where the equations integrate to the constant $T_0/2$. Note that the integral over one period of the multiplication of one sinusoid by a sinusoid of another frequency is identically zero. With these tools we can now solve for the Fourier coefficients.

To solve for the a_0 term of Equation C-4 integrate both sides of this equation with respect to time over one period. Every term within the summation will integrate to zero and we will have

$$\int_{t_0}^{t_0+T_0} x(t)dt = \int_{t_0}^{t_0+T_0} a_0 \; dt. \tag{C-11}$$

Integrating the right side and dividing both sides by T_0,

$$a_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)dt. \tag{C-12}$$

Notice that Equation C-12 is the integral of the original signal, x(t), over one period divided by the period. This is the definition for the average of x(t), which we normally call the DC value of x(t).

If we now multiply both sides of Equation C-4 by cos $(m\omega_0 t)$, where m can assume any value $-\infty \leq m \leq \infty$, and integrate this new equation over one period we get

$$\begin{aligned}\int_{t_0}^{t_0+T_0} x(t) \cos (n\omega_0 t) \; dt &= \int_{t_0}^{t_0+T_0} a_0 \cos (n\omega_0 t) \; dt + \int_{t_0}^{t_0+T_0} a_n \cos^2(n\omega_0 t) \; dt \\ &+ \int_{t_0}^{t_0+T_0} b_n \sin (n\omega_0 t) \cos (n\omega_0 t) \; dt,\end{aligned} \tag{C-13}$$

where the summation has been eliminated since all terms where m ≠ n are identically zero from Equations C-6 – C-10. Every term on the right side of Equation C-13 is zero except for the $\cos^2$ term which, using Equation C-10, integrates to $a_n\, T_0/2$. The a_n term is then

$$a_n = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\ \cos\ (n\omega_0 t)\ dt. \tag{C-14}$$

Similarly, multiplying Equation C-4 by sin $(m\omega_0 t)$ and integrating over one period yields

$$b_n = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\ \sin\ (n\omega_0 t)\ dt. \tag{C-15}$$

As an example, suppose that x(t) is a square wave of amplitude A and period T as shown in Figure C-1 below. Solve for the Fourier coefficients.

Defining t_0 as $-T_0/2$, we first solve for a_0 and get

$$a_0 = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\ dt = \frac{A}{2} \tag{C-16}$$

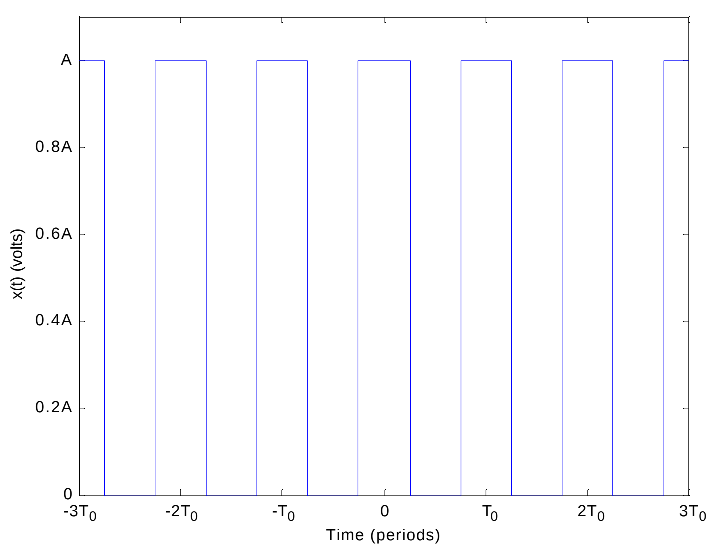

Figure C-1. Square wave of amplitude A and period T_0.

which is the DC value of x(t). Solving now for a_n with Equation C-14

$$a_n = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A \cos(n\omega_0 t)\, dt = \left[\frac{2A}{T_0 n\omega_0} \sin(n\omega_0 t)\right]_{-T_0/4}^{T_0/4}$$

$$= \frac{2A}{n\pi} \sin\frac{n\pi}{2}. \tag{C-17}$$

Notice that $a_n = 0$ for n even. Similarly, using Equation C-15 we find that $b_n = 0$.

The result of Equation C-17 is of the form sin $(\pi x)/\pi x$. (Recall that by using L'Hospital's rule $\sin(0)/0 = 1$). We will see this arise over and over so give it a special name, sinc x. Using Equation C-17 as an example, we can manipulate it into the sinc form as follows:

$$\frac{2A}{n\pi} \sin \frac{n\pi}{2} = A \frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} = A \; sinc \left(\frac{n}{2}\right). \tag{C-18}$$

A plot of sinc(x) versus x is shown in Figure C-2 below.

Using the results of the foregoing, we see that for the square wave of the example

$$x(t) = \frac{A}{2} + A \sum_{n=1}^{\infty} sinc\left(\frac{n}{2}\right) \cos(n\omega_0 t). \tag{C-19}$$

There are no $\sin(n\omega_0 t)$ terms because $b_n = 0$ for all n. We now have the spectral content of x (t). There is a DC term (f = 0), a term at the fundamental frequency, f_0, and terms at an

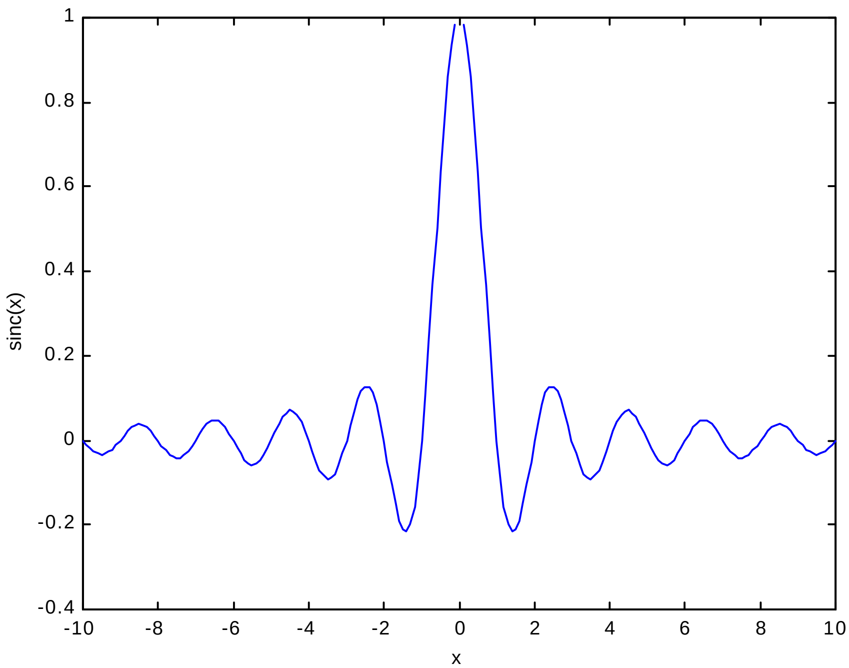

Figure C-2. Sinc(x) versus x.

infinite number of harmonics of the fundamental frequency. It is interesting to note that we started with a signal which was continuous for all time, obtained coefficients for the frequency components, and obtained a signal which is defined at only discrete values of frequency in the frequency domain.

We can plot Equation C-19 to see if it is in fact a correct representation for the square wave. We cannot sum an infinite number of terms, but the Figure C-3 shows the resultswith n = 1, 3, 5, 9, 51, and 1001. You can see that after just a few terms, the summation of cosine waves described by Equation C-19 begins converging to the square wave.

The intuitive problem with this result is that we have a square wave being represented by a combination of cosine waves. We think that this should not be possible, and in fact is not a perfect square wave unless we add all the terms out to infinity. Notice that the approximation to a square wave gets better and better as we add more and more terms to the summation.

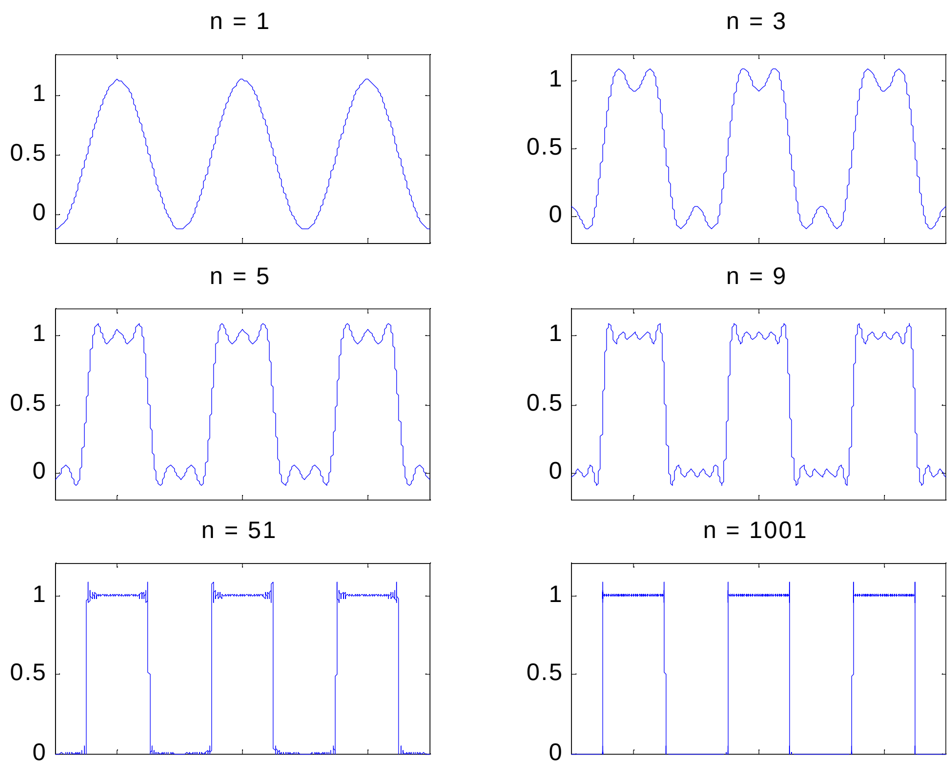

Figure C-3. Fourier series synthesis of square wave. Pictured are the series results with n = 1, 3, 5, 9, 51, and 1001.

C.2.2 Exponential form of Fourier Series

It is often beneficial to express the Fourier series in terms of a complex exponential rather than in terms of the sines and cosines. We know that sines and cosines can be expressed as complex exponentials using Euler's identity (see Section C.2.3). The *exponential Fourier series* is given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t} \tag{C-20}$$

where

$$c_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \; e^{-jn\omega_0 t} \; dt. \tag{C-21}$$

In Equation C-19 we described the square wave of x(t) as a trigonometric Fourier series. Let's use Equations C-20 and C-21 to describe the same signal using an exponential series. From C-21 we get

$$\begin{aligned} c_n &= \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} A \; e^{-jn\omega_0 t} \; dt = \left[\frac{-A}{jT_0 n\omega_0} \; e^{-jn\omega_0 t} \right]_{-T/4}^{T/4} \\ &= -\frac{A}{j2n\pi} \left[e^{-j\frac{n\pi}{2}} - e^{j\frac{n\pi}{2}} \right] = \frac{A}{n\pi} \sin\frac{n\pi}{2} = \frac{A}{2} \; sinc\left(\frac{n}{2}\right). \end{aligned} \tag{C-22}$$

Now, from C-20 we see that

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{A}{2} \; sinc\left(\frac{n}{2}\right) \; e^{jn\omega_0 t}. \tag{C-23}$$

Comparing Equation C-23 with C-19, we see that we now have negative frequencies when n is negative. The DC term is defined by n = 0, so that plugging n = 0 into Equation C-23 we get A/2 just as in the trigonometric Fourier series (since sinc(0) = 1). Because the cosine and sine consist of both positive and negative frequencies (represented by positive and negative exponentials in Euler’s identity), we see that by collecting the positive and negative terms (e.g., the fundamental frequency is represented by + and – n) Equation C-23 is identical to Equation C-19.

C.2.3 Euler's Identity

To help clarify the concept of negative frequencies introduced in the last section, we will quickly review Euler's Identity. Recall that Euler's Identity is

$$e^{j\theta} = \cos(\theta) + j\sin(\theta). \tag{C-24}$$

If we let $\theta = -\theta$, Equation C-24 becomes

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta), \tag{C-25}$$

since the cosine is an even function and the sine is an odd function (described below). We can add Equations C-24 and C-25 to get

$$\begin{array}{rl} & e^{j\theta} = \cos(\theta) + j\sin(\theta) \\ + & \underline{e^{-j\theta} = \cos(\theta) - j\sin(\theta)} \\ & e^{j\theta} + e^{-j\theta} = 2\cos(\theta) \end{array} \tag{C-26}$$

or

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}. \tag{C-27}$$

Subtracting Equation C-25 from C-24

$$\begin{array}{rl} & e^{j\theta} = \cos(\theta) + j\sin(\theta) \\ - & \underline{e^{-j\theta} = \cos(\theta) - j\sin(\theta)} \\ & e^{j\theta} - e^{-j\theta} = j2\sin(\theta) \end{array} \tag{C-28}$$

so that

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \tag{C-29}$$

Since the angle, θ, is equal to ωt, we see in Equations C-27 and C-29 that the cosine and sine are indeed defined with positive and negative frequencies. This mathematical model of the sine and cosine justifies the use of positive and negative frequencies in Equation C-23.

C.2.4 Effect of symmetry

Whenever the signal to be expanded by a Fourier series is an even function of t, that is, $x(-t) = x(t)$, only cosine terms (and possibly DC) are present (or, $b_n = 0$). This was demonstrated in Equation C-19, where only a_n terms are present. Similarly, if the signal is an odd function of t, i.e., $x(-t) = -x(t)$, then only sine terms will be present in the Fourier series ($a_n = 0$). Also, for both cases, the values of the Fourier coefficients can be obtained by integrating over half the period and doubling the result. If we are using the exponential Fourier series, the coefficients are purely imaginary for odd functions and purely real for even functions.

C.2.5 Differentiation and Integration of Fourier series

In Equation C-23 we saw that x(t) can be represented in exponential form as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t}. \tag{C-30}$$

To find the derivative of x(t) with respect to time we differentiate both sides of Equation C-30 to get

$$\frac{d}{dt}\,x(t) \;=\; \frac{d}{dt}\sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t} \;=\; \sum_{n=-\infty}^{\infty} c_n\, \frac{d}{dt}\, e^{jn\omega_0 t}. \tag{C-31}$$

We can move the derivative operator inside the summation since it is respect to t and the summation is over n. The coefficient c_n is unaffected by the derivative since it is a function of n alone and not a function of t. Differentiating term by term, Equation C-31 becomes

$$\frac{dx}{dt} \;=\; \sum_{n=-\infty}^{\infty} jn\omega_0\, c_n\, e^{jn\omega_0 t}. \tag{C-32}$$

The effect of differentiating x(t) is to multiply the Fourier coefficients by $jn\omega_0$. This has the consequence of emphasizing the higher frequencies (when n is higher) over the lower frequencies. (In other words differentiation produces a high-pass filter.) It can be easily see then that the integral of x(t) will be

$$\int x(t)dt \;=\; \sum_{n=-\infty}^{\infty} \frac{c_n}{jn\omega_0}\, e^{jn\omega_0 t} \tag{C-33}$$

which has the effect of de-emphasizing the higher frequencies (or a low-pass filter).

C.2.6 Effect of time shift of signal

We already have seen that we can represent a time-domain signal as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t}. \tag{C-34}$$

What if we delay this signal so that x(t) becomes x(t–τ)? To find out, substitute t–τ for t in Equation C-34 to get

$$x(t-\tau) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 (t-\tau)} = \sum_{n=-\infty}^{\infty} \left[c_n \, e^{-jn\omega_0 \tau} \right] e^{jn\omega_0 t}. \tag{C-35}$$

You can see that we get the same Fourier coefficients, c_n, but they have shifted in phase by $e^{jn\omega_0\tau}$. In other words, delaying the signal by τ seconds causes a change in the phase of the Fourier series by $n\omega_0\tau$ radians.

C.2.7 Convergence of the Fourier series

We mentioned earlier that the Fourier series will converge to x(t) when certain conditions are met. The conditions are known as Dirichlet conditions and are met with all real systems, which are the type that in which we have interest. This means that any communications signal we encounter will converge in a Fourier series. There are three Dirichlet conditions to be met, which are 1) The function or signal is absolutely integrable over its period T_0 (i.e., the integral does not go to infinity), 2) There are a finite number of maxima and minima over the period, and 3) There are a finite number of discontinuities over the period.

C.2.8 Spectrum of a periodic signal

We now know that we can expand a periodic signal into DC and its frequency components using the Fourier series. We can therefore plot the magnitude of its frequency components to develop the spectrum of the signal. This can be easily accomplished by plotting c_n versus f (or ω) as shown in Figure C-4 from Equation C-19 for the square wave

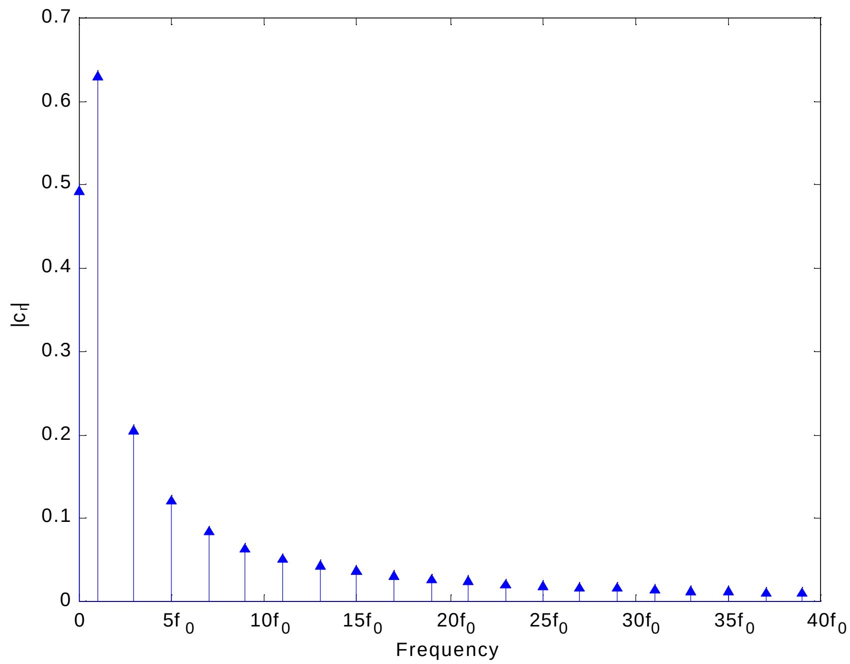

Figure C-4. $|c_n|$ versus frequency for square wave of Figure C-1.

of Figure C-1. We see that we have values for DC, the fundamental frequency f_0, and all of its harmonics--sometimes the value of a harmonic is zero. We only see the positive frequencies in this figure, because it is a plot of a trigonometric series. If it were from an exponential series, there would also be negative frequencies whose amplitudes would be the conjugate of the amplitudes of the positive frequencies. Because there are discrete lines to represent amplitudes only at discrete frequencies, this called a discrete spectrum.

Notice that in the figure that all the magnitudes are positive. Since the amplitudes of the individual components of Equation C-19 do include negative values, this discrepancy is resolved by representing the negative values in the phase (phase plot not shown). Since a phase change of 180 degrees is indicative of a negation, those areas of the spectrum where the amplitude is actually negative in magnitude would have a phase of 180 degrees.

C.2.9 Average power of a periodic signal

The average power in a periodic signal, x(t) is given by

$$P = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x^2(t)\ dt, \tag{C-36}$$

which is just the integral over one period of the square of the signal divided by the period. This is the normalized power developed across a one-ohm resistor. Now that we have the Fourier series representation of this same signal, how do we determine the average power contained in the series?

We know that, of course, the average power is the same no matter how we represent the signal. Recall from Equation C-20 that

$$x(t) = \sum_{n=-\infty}^{\infty} c_n\ e^{jn\omega_0 t}. \tag{C-37}$$

Squaring both sides

$$x^2(t) = \sum_{n=-\infty}^{\infty} |c_n|^2\ |e^{jn\omega_0 t}|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{C-38}$$

Using Equations C-36 and C-38,

$$P = \frac{1}{T_0}\int_0^{T_0} x^2(t)\ dt = \frac{1}{T_0}\sum_{n=-\infty}^{\infty}\int_0^{T_0} |c_n|^2\ dt = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{C-39}$$

This is intuitively satisfying since it tells us that all the power of the signal is contained in the amplitude values of the spectrum of the signal.

These relationships, of the power contained in the signal x(t), being identical to the power contained in the spectrum representation of the same signal, is known as Parseval's theorem. These relationships apply to power signals (as opposed to energy signals where the average power is zero).

C.3 FOURIER TRANSFORMS

In the last sections we have developed the Fourier series which we saw describes a periodic signal. With the Fourier series we are able to determine the frequency content of a periodic signal. We now want to develop a method to determine the frequency or spectral content of a nonperiodic signal. This method is the *Fourier transform* which we will now derive from the Fourier series.

C.3.1 Transform Derivation

Using the exponential form of the Fourier series we know from Equation C-20 that

$$x(t) = \sum_{n=-\infty}^{\infty} c_n\ e^{jn\omega_0 t} \tag{C-40}$$

where

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\ e^{-jn\omega_0 t}\ dt. \tag{C-41}$$

If we have a nonperiodic signal, then the period of x(t) approaches infinity, that is, $T_0 \rightarrow \infty$. Since $\omega_0 = 2\pi/T_0$, as $T_0 \rightarrow \infty$, $\omega_0 \rightarrow d\omega$, which causes ω to become a continuous variable (rather than discrete as it was for a periodic signal) and the quantity $n\omega_0$ to approach $nd\omega$, i.e., $n\omega_0 \rightarrow \omega$ as $n \rightarrow \infty$. Making these substitutions into Equation C-41 for a nonperiodic signal we get

$$c_n = \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} x(t)\ e^{-j\omega t}\ dt. \tag{C-42}$$

We can see that t is integrated out, so that the result of the integral is a function of ω alone. This integral is now a function of frequency, and since it was generated from x(t), we will capitalize the x (to signify that it is a frequency function), and call this new function X(ω), i.e.,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\ e^{-j\omega t}\ dt. \tag{C-43}$$

Equation C-42 now becomes

$$c_n = \frac{d\omega}{2\pi} X(\omega). \tag{C-44}$$

Substituting Equation C-44 into C-40, we get

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{d\omega}{2\pi} X(\omega) e^{jn\omega_0 t}. \tag{C-45}$$

Recognizing that we are now summing a continuous variable, as $\omega_0 \rightarrow d\omega$ and $n\omega_0 \rightarrow \omega$, the infinite summation becomes an integral, giving us

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega. \tag{C-46}$$

Equation C-43 is known as the *Fourier integral* or the *Fourier transform* of x(t). The variable t has disappeared from the equation and the transformed variable X is a function of frequency. Equation C-46 is the inverse Fourier transform and transforms a frequency domain signal into its time domain counterpart. The functions x(t) and X(ω) make up a Fourier pair which is often seen as $x(t) \rightleftarrows X(\omega)$. By convention, the capital X is reserved for frequency domain signals while the lower case x is used for the time domain signals. This is true no matter what letter is used, e.g., $u(t) \rightleftarrows U(\omega)$. Another convention that is used is that the Fourier transform of x(t) is $\mathscr{F}[x(t)]$, that is, $X(\omega) = \mathscr{F}[x(t)]$. Similarly, we can write $x(t) = \mathscr{F}^{-1}[X(\omega)]$.

Just as for the Fourier series, the Fourier transform will only exist if x(t) satisfies the Dirichlet conditions.

We can define the Fourier transform in terms of frequency f, instead of angular frequency ω. One advantage of using the f frequency is that the 2π constant goes away. The f Fourier relationships are given by

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df, \quad and$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt. \tag{C-47}$$

Consider the waveform shown in Figure C-5. Here we have a single pulse of magnitude A, centered at 0, with width T, i.e., x(t) = A rect(t/T). This is a very important signal in the world of communications and radar. This is also the pulse shape of clock pulses

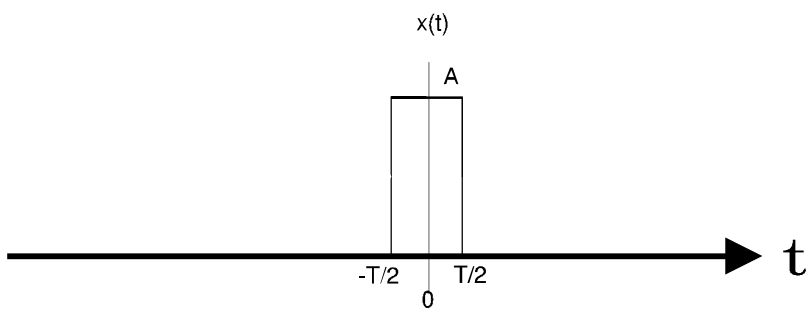

Figure C-5. Square pulse of amplitude A and duration T.

for the computer world. Using Equation C-43 to find the Fourier transform of the pulse shown in the figure

$$X(\omega) = \int_{-T/2}^{T/2} Ae^{-j\omega t}dt = \left[\frac{-A}{j\omega}\, e^{-j\omega t}\right]_{-T/2}^{T/2} = \frac{-A}{j\omega}\left[e^{-\frac{j\omega T}{2}} - e^{\frac{j\omega T}{2}}\right] = \frac{2A}{\omega}\, \sin\left(\frac{\omega T}{2}\right). \tag{C-48}$$

If we make the substitution of $\omega = 2\pi f$, (or compute the transform directly from Equation C-47) we find X(f) as

$$X(f) = \frac{2A}{\omega} \sin\left(\frac{\omega T}{2}\right) = \frac{A}{\pi f} \sin(\pi f T) = AT \frac{\sin(\pi f T)}{\pi f T} = AT \; sinc \; (fT). \tag{C-49}$$

Therefore, any time you see a square wave pulse (whether periodic or non periodic), think sinc. Likewise, we will see (through the principle of duality) that a square pulse in the frequency domain will require a sinc pulse in the time domain. Using the Fourier pair convention then

$$A \; rect\left(\frac{t}{T}\right) \rightleftarrows AT \; sinc \; (fT). \tag{C-50}$$

Notice that the left side of this equation is completely a function of time while the right side is only a function of frequency.

C.3.2 Magnitude and phase of the Fourier transform

Just a few words about what the complex representation of a signal in the frequency domain means. If a signal is represented by a complex number then it has a phase other than zero. If its representation is purely real, then its phase is zero, and if its representation is purely imaginary, then its phase is $\pm\ 90°$.

Take the Fourier transform X(ω) which is, in general, a complex variable. In polar form this function consists of a magnitude, depicted as |X(ω)|, and phase ϕ represented by $e^{j\phi} = e^{j \arctan[Im(X(\omega)) / Re(X(\omega))]}$, where Im is the imaginary part and Re is the real part of X(ω). Therefore,

$$X(\omega) = |X(\omega)| e^{j \tan^{-1}\left[\frac{Im(X(\omega))}{Re(X(\omega))}\right]}. \tag{C-51}$$

From this representation you can see that if the real part is zero, the arctangent of infinity is 90 degrees as we expect, while if the imaginary part is zero, then the phase is zero.

As an example of a function which can be represented by Equation C-51 consider the negative exponential function

$$x(t) = e^{-at}u(t), \tag{C-52}$$

where u(t) is the unit step. We can find the Fourier transform as

$$\begin{aligned} X(\omega) &= \int_0^\infty e^{-at}e^{-j\omega t}\, dt = \int_0^\infty e^{-(a+j\omega)t}\, dt \\ &= \frac{-1}{a + j\omega}\, e^{-(a + j\omega)t}\Big|_0^\infty = \frac{1}{a + j\omega}. \end{aligned} \tag{C-53}$$

Recall that the magnitude of a complex number is the square root of that number multiplied by its conjugate. The complex part of Equation C-53 multiplied by its conjugate is found as $(a + j\omega)(a - j\omega)$ which equals $a^2 + \omega^2$. The magnitude of $X(\omega)$ is the square root of this or $|X(\omega)| = 1/(a^2 + \omega^2)^{1/2}$. The phase of $X(\omega)$ will be negative (since the complex number is in the denominator) and will be $-\tan^{-1}[\omega/a]$. We can see then that $X(\omega)$ of Equation C-53 can equivalently be represented as

$$X(\omega) = \frac{1}{\sqrt{a^2 + \omega^2}}\, e^{-j\tan^{-1}\left[\frac{\omega}{a}\right]}. \tag{C-54}$$

This transformation is simply converting from the rectangular representation to the polar representation. If your calculator performs this function, you can easily compute the magnitude and phase of any complex signal.

C.3.3 Properties of the Fourier Transform

It is often undesirable or very difficult to perform the transformation of a signal from the time domain to the frequency domain or from the frequency domain to the time domain. We can often use properties of the F.T. which allow us to recognize what the transform will be without having to perform the mathematical transformation. Some of these properties are described below.

C.3.3.1 Linearity

The first property we want to look at is linearity. If the transform is linear, and it is, then superposition applies. The implication is that if $x_1(t) \rightleftharpoons X_1(\omega)$ and $x_2(t) \rightleftharpoons X_2(\omega)$, then

$$\mathscr{F}[a_1 x_1(t) + a_2 x_2(t)] = a_1 X_1(\omega) + a_2 X_2(\omega). \tag{C-55}$$

C.3.3.2 Duality

Another property which is a powerful tool is that of duality. The essence of duality is that if we know a transform, e.g., a Fourier transform of x(t), then if we encounter a function which is identical in form to x(t) but is in the frequency domain, then duality tells us what the inverse transform of this frequency signal will be. In mathematical terms the duality property states simply that if $x(t) \rightleftharpoons X(\omega)$ then

$$X(t) \rightleftarrows 2\pi\ x(-\omega) \quad or$$
$$X(t) \rightleftarrows x(-f). \tag{C-56}$$

For example, if x(t) = rect(t), then X(f) = sinc(f) as we found in Equation C-49. Now, if x(t) = sinc(t), what is X(f)? It is very difficult to perform the Fourier integration on the sinc function. How then do we determine the transform? We find it is easier to use duality. We have as a Fourier pair

$$rect(t) \rightleftarrows sinc(f). \tag{C-57}$$

We recognize sinc(t) as the dual of X(f) (i.e., the right side of the equation). We let

$$X(t) = sinc(t), \tag{C-58}$$

where we use X(t) (capital X) to signify that we are applying the dual. From Equation C-56 we substitute –f for t in Equation C-57, to give

$$sinc(t) \rightleftarrows rect(-f). \tag{C-59}$$

Examination of the rect function shows that it is an even function so that

$$sinc(t) \rightleftarrows rect(f). \tag{C-60}$$

C.3.3.3 Differentiation and Integration

We saw in Fourier series that differentiation and integration of x(t) in the time domain implied multiplication or division by a complex constant in the frequency domain. The same rules follow for the transform and if $x(t) \rightleftarrows X(\omega)$ then

$$x'(t) \rightleftarrows j\omega X(\omega), \quad and$$
$$\int x(t)dt \rightleftarrows \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega). \tag{C-61}$$

Notice that the constant of integration appears as $\pi X(0)\delta(\omega)$.

C.3.3.4 Time and Frequency Scaling

The next property is that of scaling, both time and frequency. Suppose you know the transform of x(t), but now you have x(3t). What is the transform? The property is (which can be used whether the signal has been time scaled or frequency scaled), if $x(t) \rightleftarrows X(\omega)$ then

$$x(at) \rightleftarrows \frac{1}{|a|} X\left(\frac{\omega}{a}\right). \tag{C-62}$$

C.3.3.5 Time Shifting

We saw in the Fourier series (Equation C-35) that if we delayed x(t) to $x(t-\tau)$, time shifting, that the phase of the frequency domain representation changed in direct proportion to τ. The same is true for the F.T. If $x(t) \rightleftarrows X(\omega)$ then

$$x(t - \tau) \rightleftarrows e^{-j\omega\tau}X(\omega). \tag{C-63}$$

C.3.3.6 Frequency Shifting

Now for some frequency domain properties. We had time shifting now let's see what happens with frequency shifting. If $X(\omega) \rightleftarrows x(t)$ then

$$X(\omega - \omega_0) \rightleftarrows e^{j\omega_0 t}\, x(t). \tag{C-64}$$

Another name for frequency shifting is modulation. We see that if

$$e^{j\omega_0 t}\, x(t) \rightleftarrows X(\omega - \omega_0), \tag{C-65}$$

and

$$\cos(\omega_0 t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right], \tag{C-66}$$

then

$$\cos(\omega_0 t)\, x(t) \rightleftarrows \frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]. \tag{C-67}$$

Thus, modulation of a cosine wave by a time function x(t) results in a new function having a spectrum consisting of half the original spectrum of x(t) translated along the positive frequency axis by an amount ω_0 and the other half translated along the negative frequency axis by an amount $-\omega_0$.

C.3.3.7 Convolution

Convolution is the integral which allows us to find the response of a system, y(t), to an arbitrary input, x(t), if we know the unit impulse response, h(t) of the system. Mathematically, we can find the output as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau. \tag{C-68}$$

Let's find the Fourier transform of y(t), given by

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\ e^{-j\omega t}\ dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau\right] e^{-j\omega t}\ dt. \tag{C-69}$$

Interchanging the order of integration and integrating first with respect to t

$$Y(\omega) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\ e^{-j\omega t} dt\right] d\tau. \tag{C-70}$$

Evaluating the inner integral first we recognize it as the F.T. of h(t–τ). From Equation C-63 this integral is found as

$$\mathscr{F}[h(t-\tau)] = e^{-j\omega\tau}\ H(\omega). \tag{C-71}$$

Replacing the inner integral with Equation C-71, Equation C-70 becomes

$$Y(\omega) = \int_{-\infty}^{\infty} x(\tau)\ e^{-j\omega\tau} H(\omega)\ d\tau = H(\omega) \int_{-\infty}^{\infty} x(\tau)\ e^{-j\omega\tau}\ d\tau, \tag{C-72}$$

where we have removed H(ω) from within the integral as it is a constant with respect to the integration. (The function H(ω) is the transfer function of the system and is the Fourier transform of the impulse response h(t).) The remaining integral is just the F.T. of x(t), (i.e., X(ω)) with τ replacing t for the integration so that

$$Y(\omega) = H(\omega)\ X(\omega). \tag{C-73}$$

Using the same procedure in the cyclic frequency domain we find

$$Y(f) = H(f)\ X(f). \tag{C-74}$$

From this discovery we conclude that convolution in the time domain is equivalent to multiplication is the frequency domain. First we find the Fourier transforms of x(t) and h(t), obtaining X(ω) and H(ω), and multiply them together. Knowing Y(ω) we can find y(t) by finding the inverse Fourier transform of Y(ω). It is this powerful property of the frequency domain that makes it so useful.

C.4 FOURIER TRANSFORMS OF PERIODIC FUNCTIONS (POWER SIGNALS)

Now that we have this powerful tool of the Fourier transform whereby we can leave the time domain and enter the frequency domain for nonperiodic signals (energy signals), how to we expand this concept to include the periodic signals? The answer can be found in the inverse F.T. of the delta function. Let's find the inverse F.T. of a frequency-shifted delta function, $\delta(f - f_0)$. Going back to the defining equation

$$\mathscr{F}^{-1}\ [\delta(f - f_0)] = \int_{-\infty}^{\infty} [\delta(f - f_0)]\ e^{j2\pi ft}\ df. \tag{C-75}$$

Using the sifting property of the delta function the integral is simply $\exp(j2\pi f_0 t)$, so that we get a new pair

$$e^{j2\pi f_0 t} \rightleftarrows \delta(f - f_0). \tag{C-76}$$

Because we can describe sines and cosines in terms of exponentials, we now have the F.T. for these sinusoids. It is easily seen that

$$\cos(2\pi f_0 t) = \frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right] \rightleftarrows \frac{1}{2}[\delta(f - f_0) + (f + f_0)] \tag{C-77}$$

and

$$\sin(2\pi f_0 t) = \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right] \rightleftarrows \frac{1}{2j}\left[\delta(f - f_0) - (f + f_0)\right]. \tag{C-78}$$

Extending this concept, since using Fourier techniques we can describe any periodic function as a sum of sines and cosines, we can now transform any periodic function which conforms to the Dirichlet conditions.

C.5 ENERGY CONTENT OF A SIGNAL

We saw that the power contained in a periodic signal is the same whether we determine the power from the time domain representation, or the frequency content representation of the Fourier series. Similarly, the energy of a nonperiodic signal must be the same whether we derive it from the time domain or from the frequency domain of the signal.

Recall that the energy E of a signal x(t) is the integral of $x^2(t)$ over all time. Since $x^2(t) = x(t)x(t)$, we simply let the second x(t) be represented by the inverse F.T. of $X(\omega)$ and perform the integration, i.e.,

$$E = \int_{-\infty}^{\infty} x^2(t)\ dt = \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} X(f)\ e^{j2\pi ft} df\right] dt. \tag{C-79}$$

Rearranging the order of integration

$$E = \int_{-\infty}^{\infty} X(f) \left[\int_{-\infty}^{\infty} x(t)\ e^{j2\pi ft} dt\right] df = \int_{-\infty}^{\infty} X(f)\ X^*(f)\ df, \tag{C-80}$$

where X*(f) is the complex conjugate of X(f). Therefore, we find that the energy in the nonperiodic signal as

$$E = \int_{-\infty}^{\infty} x^2(t)\ dt = \int_{-\infty}^{\infty} |X(f)|^2\ df. \tag{C-81}$$

This is the generalized form of Parseval's theorem for nonperiodic signals.